

EU-Workshop Series on Mathematical Optimization Models for Financial Institutions

“The Impact of Volatility Estimates in Hedging Effectiveness”

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Overview

- **Introduction**
- **Brief Literature Review**
- **Application**
- **Conclusions**

Introduction

- One of the basic inputs in most option pricing formulas is the volatility (StDev) of the underlying security
- Errors in the estimation of the volatility lead to two kinds of errors in trading options
 - The fair value of the option
 - The corresponding replicating portfolio

- **This class of problems belongs to the area of estimation risk**
- **It is part of a broader research area called model risk**
- **We examine the effect of hedging European contingent claims with a volatility that differs from the true one**

Literature Review

- **There is a broad literature examining the effect of sampling errors in the variance estimates**
- **For European option prices a substantial complication arises from the fact that even an unbiased estimate of the variance does not produce an unbiased estimate of the option price, since the option pricing formulas are non-linear with respect to variance**

Literature Review

- **Lo (1986)** argued that since parameter estimates are ultimately employed in the pricing formulas in place of the true but unknown parameters, the sampling variation of parameter estimates will induce sampling variation in the estimated contingent claim prices about their true values
- The variance can be mis-specified from the omission of discontinuities (Merton, 1977) or the misspecification of the drift in the data generating process (Lo and Wang, 1995)

Literature Review

- **Figlewski (1987)** argues that the effect of volatility mis-estimation is larger for out of the money call options
- **Avellaneda *et al* (1995)**, derive pricing and hedging bounds when the uncertain volatility is confined in a minimum and maximum bound
- **El Karoui *et al* (1998)** show that under deterministic volatilities the effect of mis-estimation can be positive or negative
- **Anagnou and Hodges (2002)** examine the hedging errors under mis-estimated volatility in a discrete framework , they find that the hedging errors are more pronounced for at the money options

Assumptions

- **Frictionless Markets**
- **The evolution of the underlying security is governed by a geometric Brownian motion with constant volatility**

$$dS(t) = S(t) [\mu dt + \sigma dW(t)]$$

- **Zero risk-free interest rates**
- **We sell a Call option priced at the wrong volatility and we hedge it until expiration**

- We define the tracking error as the difference between the replicating portfolio and the option price given by the true volatility

$$e_t \equiv \Pi_t - C_t$$

- Using standard rules from stochastic calculus the tracking error at expiration is given by

$$e_T = (\Pi_0 - C_0) + \int_0^T (\hat{\Delta}_u - \Delta_u) \mu S_u du + \int_0^T (\hat{\Delta}_u - \Delta_u) \sigma S_u dW_u$$

- **The first term in the tracking error is the difference between the true premium and the estimated premium and takes place at the initiation of the trade**
- **The second term is attributed to the difference between the delta with the true volatility as input and the delta obtained with the estimated volatility**
- **The effect of this difference also depends on the excess return over the risk free rate namely the market price of risk**
- **The third term is ascribed again to the difference between the two deltas but in addition depends on the true volatility and the Brownian noise**

Application - Black and Scholes Formula

- We exclude from the analysis the difference between the true premiums.
- We derive the expected value and the standard deviation of the DTE for arbitrary estimators of the volatility by taking a Taylor's expansion around the true volatility

$$E_0^P (\Delta e_T) = E \left\{ \int_0^T -(\sigma - \hat{\sigma}) \frac{\partial \Delta_u}{\partial \sigma} \mu S_u du \right\}$$

Application - Black and Scholes Formula

- **First Moment**

$$E_0^P (\Delta e_T) = \frac{(\sigma - \hat{\sigma})}{\sigma} \mu E \left[\int_0^T n(h_u^+) h_u^- S_u du \right]$$

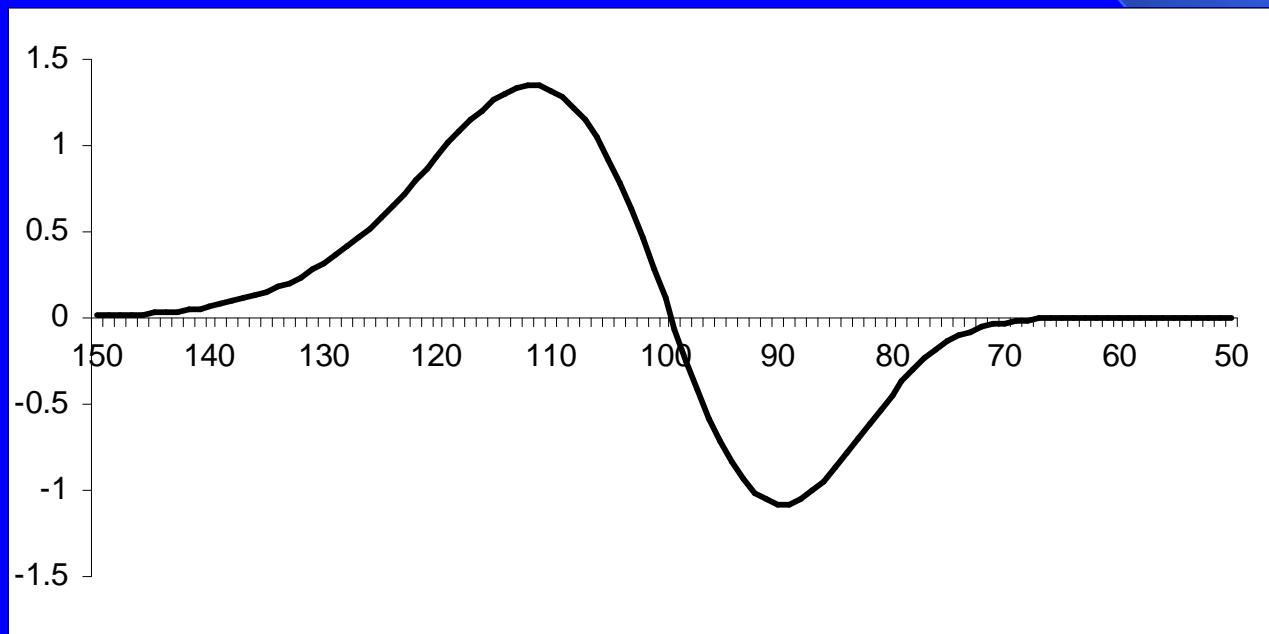
- **Second Moment**

$$E_0^P (\Delta e_T^2) = (\sigma - \hat{\sigma})^2 E \left[\int_0^T (n(h_u^+))^2 (h_u^-)^2 S_u^2 du \right]$$

$n(\cdot)$ is the standard normal probability density function.

Application - Black and Scholes Formula

- The derivative of the delta with respect to the volatility (abbreviated as delvar) across moneyness



Application - Black and Scholes Formula

- The sign of the expected DTE cannot be unambiguously determined since the sensitivity of the delta hedge ratio with respect to the volatility is neither strictly positive nor strictly negative.
- Delta is positive for out of the money options decreases as the strike price approaches the current price of the underlying and equals zero for at the money options.
- For in the money options is negative and converges to zero for deep in the money options
- It has the highest values for out of the money options

Application - Black and Scholes Formula

- **A trader using a volatility that is smaller than the true one, for out of the money options he will hedge with a delta that is smaller than the delta that governs the evolution of the call**
- **Conversely, when the spot price is above the strike the estimated delta will be larger than the true delta even though the volatility input is smaller than the true one**

Implementation

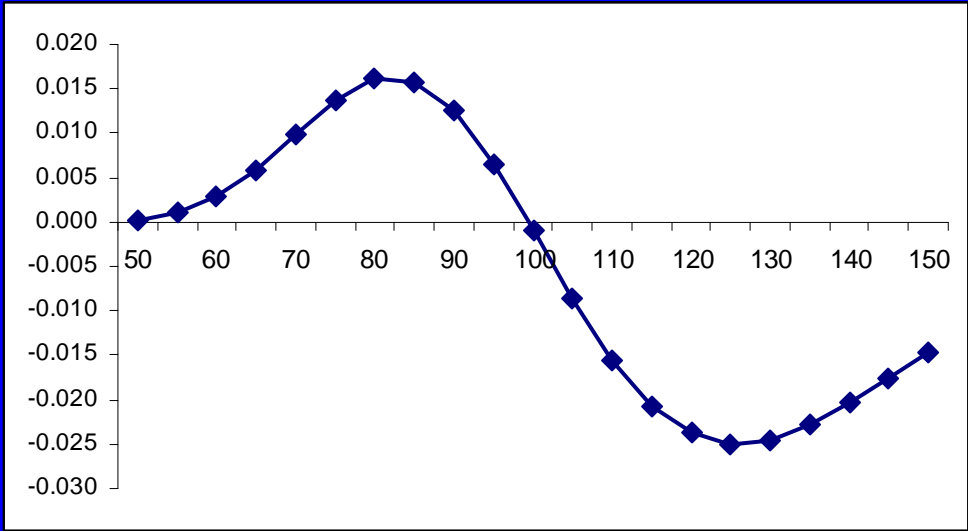
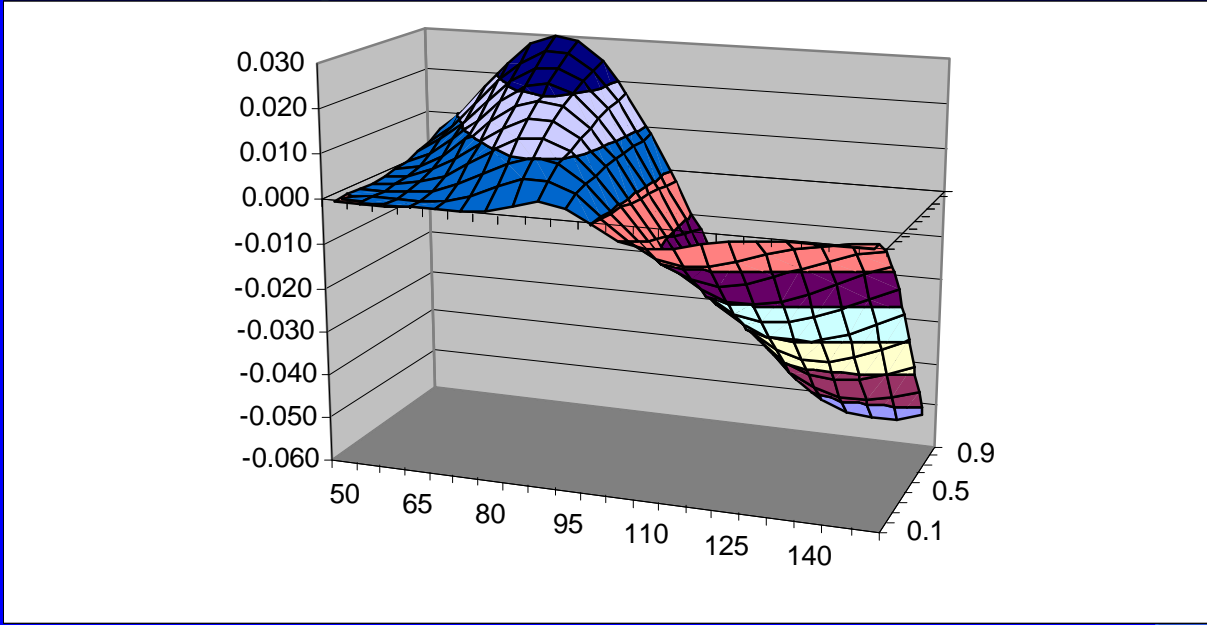
- Using some tedious algebra the two moments can be computed via numerical integration
- We compute the two moments across various maturities, true volatilities and risk premiums
- The difference between the true volatility and the estimated volatility is assumed to be 1%
- All the results are based on the assumption that the true volatility $>$ estimated volatility

Results

- **Generally the expected value of the tracking error is negative for out of the money options and positive for in the money.**
- **Nevertheless for low levels of volatility and high risk premiums options slightly out of the money and with time to maturity more than 4-months have positive expected tracking error**
- **Even though the volatility is lower at the beginning at some point the option goes in the money and the trader is compensated for the losses**

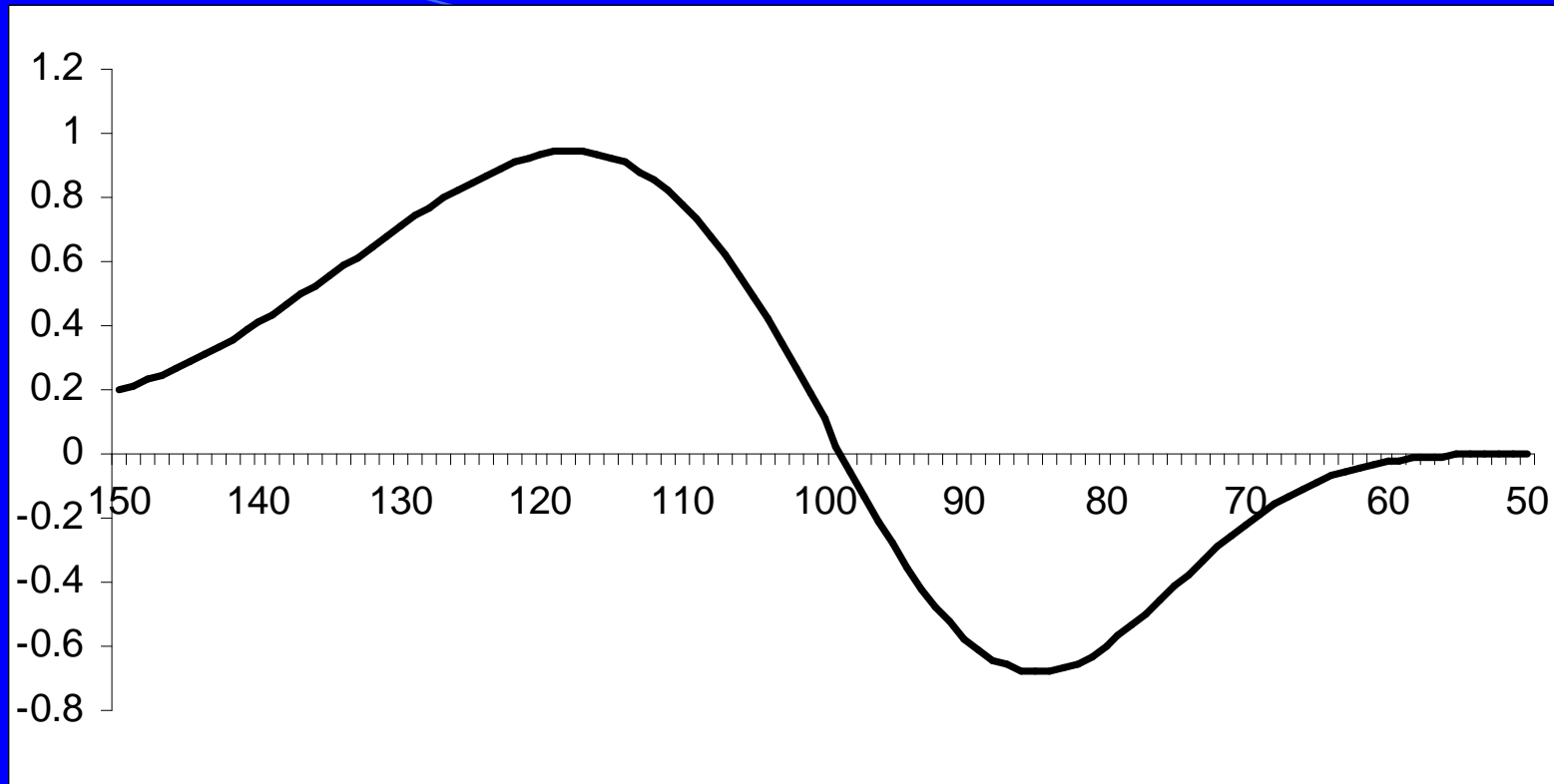
Results-risk premium 0.1 & volatility 0.3

Strike/Time	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
50	0.00000	0.00000	0.00001	0.00007	0.00029	0.00076	0.00155	0.00256	0.00407	0.00576
55	0.00000	0.00000	0.00006	0.00035	0.00105	0.00222	0.00383	0.00582	0.00811	0.01062
60	0.00000	0.00002	0.00032	0.00122	0.00280	0.00496	0.00755	0.01042	0.01347	0.01662
65	0.00000	0.00017	0.00115	0.00313	0.00583	0.00897	0.01234	0.01580	0.01926	0.02268
70	0.00002	0.00075	0.00298	0.00618	0.00980	0.01353	0.01721	0.02079	0.02422	0.02749
75	0.00016	0.00221	0.00580	0.00977	0.01368	0.01738	0.02083	0.02405	0.02705	0.02985
80	0.00081	0.00452	0.00872	0.01260	0.01607	0.01919	0.02199	0.02454	0.02687	0.02902
85	0.00226	0.00661	0.01025	0.01327	0.01583	0.01806	0.02002	0.02178	0.02338	0.02483
90	0.00359	0.00683	0.00916	0.01100	0.01252	0.01382	0.01495	0.01595	0.01684	0.01763
95	0.00290	0.00431	0.00527	0.00600	0.00658	0.00705	0.00743	0.00774	0.00799	0.00819
100	-0.00008	-0.00023	-0.00042	-0.00065	-0.00091	-0.00120	-0.00151	-0.00184	-0.00219	-0.00257
105	-0.00320	-0.00494	-0.00632	-0.00754	-0.00866	-0.00972	-0.01074	-0.01174	-0.01271	-0.01367
110	-0.00443	-0.00816	-0.01099	-0.01337	-0.01550	-0.01745	-0.01927	-0.02100	-0.02266	-0.02427
115	-0.00377	-0.00923	-0.01365	-0.01737	-0.02065	-0.02362	-0.02637	-0.02894	-0.03137	-0.03370
120	-0.00238	-0.00851	-0.01426	-0.01930	-0.02380	-0.02787	-0.03161	-0.03511	-0.03840	-0.04153
125	-0.00120	-0.00680	-0.01327	-0.01938	-0.02498	-0.03013	-0.03490	-0.03936	-0.04356	-0.04754
130	-0.00051	-0.00487	-0.01133	-0.01805	-0.02452	-0.03061	-0.03634	-0.04173	-0.04683	-0.05167
135	-0.00018	-0.00319	-0.00903	-0.01587	-0.02284	-0.02965	-0.03618	-0.04241	-0.04836	-0.05403
140	-0.00006	-0.00194	-0.00680	-0.01330	-0.02042	-0.02765	-0.03477	-0.04169	-0.04837	-0.05481
145	-0.00002	-0.00111	-0.00489	-0.01072	-0.01762	-0.02498	-0.03246	-0.03987	-0.04714	-0.05423
150	0.00000	-0.00060	-0.00337	-0.00836	-0.01478	-0.02199	-0.02957	-0.03728	-0.04497	-0.05258



Results

- For the same level of risk premium and increasing volatility the tracking error for out of the money options and in the money options increases with the exception of long term out of the money options where it decreases slightly
- Ceteris Paribus the tracking error is an increasing function of time



Effect in delvar from an increase in the volatility (0.2 to 0.3)

- **At the money options have positive tracking error for low levels of volatility and negative for high levels of volatility**

Results-Standard Deviation

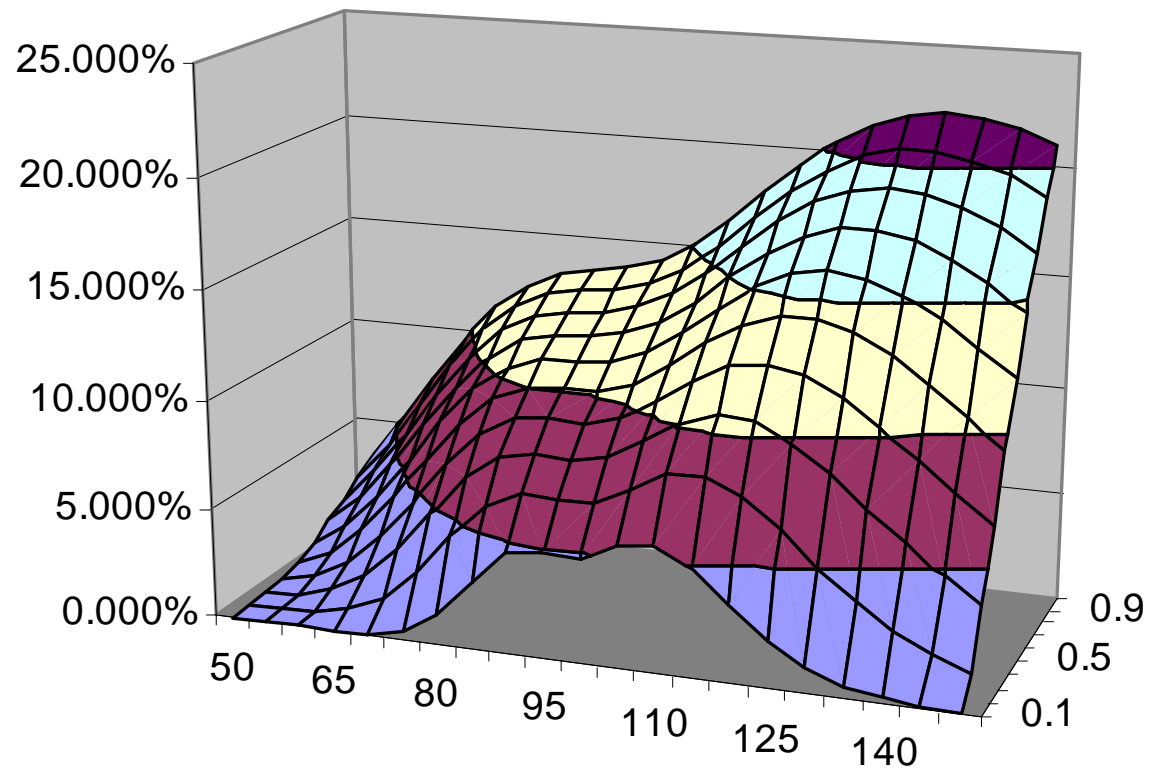
- **The standard deviation of the tracking error is an increasing function of the volatility and an increasing function of the risk premium**
- **The smaller Standard Deviation is exhibited by in the money options and the larger for out of the money options**
- **The two maxima are in regions near the price of the underlying asset (plus/minus 10%)**

Risk Premium -0.05

Volatility – 0.25

Strike/price	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
50	0.000%	0.003%	0.043%	0.185%	0.462%	0.871%	1.389%	1.985%	2.631%	3.304%
55	0.000%	0.021%	0.179%	0.563%	1.154%	1.891%	2.712%	3.568%	4.428%	5.269%
60	0.001%	0.109%	0.568%	1.372%	2.374%	3.453%	4.530%	5.568%	6.546%	7.458%
65	0.014%	0.408%	1.423%	2.744%	4.116%	5.423%	6.622%	7.706%	8.683%	9.562%
70	0.098%	1.165%	2.874%	4.598%	6.142%	7.480%	8.636%	9.639%	10.518%	11.295%
75	0.457%	2.567%	4.765%	6.575%	8.033%	9.227%	10.226%	11.081%	11.828%	12.491%
80	1.458%	4.425%	6.599%	8.173%	9.384%	10.366%	11.197%	11.921%	12.569%	13.158%
85	3.163%	6.057%	7.792%	9.031%	10.017%	10.854%	11.595%	12.268%	12.892%	13.477%
90	4.660%	6.769%	8.118%	9.189%	10.115%	10.946%	11.712%	12.427%	13.102%	13.744%
95	4.895%	6.649%	7.999%	9.149%	10.171%	11.101%	11.962%	12.768%	13.528%	14.251%
100	4.752%	6.726%	8.246%	9.530%	10.665%	11.693%	12.641%	13.526%	14.359%	15.149%
105	5.496%	7.552%	9.134%	10.477%	11.667%	12.748%	13.747%	14.680%	15.559%	16.393%
110	5.780%	8.367%	10.140%	11.587%	12.850%	13.989%	15.037%	16.014%	16.934%	17.806%
115	4.903%	8.445%	10.682%	12.393%	13.826%	15.086%	16.226%	17.278%	18.261%	19.189%
120	3.450%	7.706%	10.538%	12.645%	14.353%	15.814%	17.109%	18.283%	19.367%	20.378%
125	2.107%	6.446%	9.776%	12.314%	14.357%	16.078%	17.580%	18.922%	20.144%	21.273%
130	1.160%	5.025%	8.605%	11.503%	13.877%	15.881%	17.619%	19.161%	20.552%	21.826%
135	0.592%	3.705%	7.251%	10.370%	13.019%	15.288%	17.265%	19.017%	20.593%	22.030%
140	0.286%	2.616%	5.897%	9.073%	11.904%	14.390%	16.583%	18.539%	20.302%	21.908%
145	0.133%	1.786%	4.661%	7.745%	10.651%	13.286%	15.655%	17.791%	19.729%	21.499%
150	0.060%	1.189%	3.600%	6.478%	9.358%	12.067%	14.560%	16.842%	18.931%	20.853%

Risk Premium -0.05 Volatility – 0.25



Conclusions

- The tracking error is most pronounced for out of the money options.
- It may be appropriate for out of the money options to compute the hedge ratio using a higher volatility than what the trader expects to prevail in the future
- Uncertainty is most pronounced for long term options
- For long term options and high volatility the first order approximation can be used for larger differences between the true volatility and the estimated volatility