Tricriterion Models for Suitable Portfolio Investors

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research in cooperation with

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Agenda

1. Suitable Portfolio Investors
2. Computation of efficient portfolios for E-V-M
3. Nonlinear Criteria
1. Suitable Portfolio Investors

1.1 Portfolio selection as a stochastic optimization problem
1.2 Expected Utility Substitute Problem
1.3 Risk Aversion and the E-V Problem
1.4 Multiple Criteria
1.1 Portfolio Selection as a Stochastic PP

- **Stochastic Programming Problem:**
  Let \( x \in X \) denote a decision alternative and \( r \) a random vector. Then a stochastic optimization problem is
  \[
  \text{max} \{ R(x, r) : x \in X \}
  \]

- **Portfolio Selection (Single-Stage):**
  Let \( x_1, \ldots, x_n \) denote the proportions of an investor’s capital to be invested in \( n \) securities with future returns \( r_1, \ldots, r_n \), i.e., maximize portfolio return
  \[
  R(x, r) = \sum_{i=1}^{n} r_i x_i
  \]
  where
  \[
  X = \left\{ x \in R^n : \sum_{i=1}^{n} x_i = 1, \ x_i \geq 0 \right\}
  \]
  “max” not well-defined, since \( R(x, r) \) is a random variable.
1.2 Expected Utility Substitute Problem

- **Von Neumann and Morgenstern:** A **rational** decision maker facing uncertainty should decide to maximize expected utility.

- Let $U$ denote the decision maker’s utility function, that is, $U$ is increasing and the decision maker’s utility is $U(R(x,r))$. Then $x$ solves the stochastic optimization problem

$$\max \{ R(x,r) : x \in X \}$$

iff it solves the **substitute problem** ($E$ expected value)

$$\max \{ E [U(R(x,r))] : x \in X \}$$

- decision maker’s utility function is generally unknown!
1.3 Risk Aversion and the E-V Problem

- **Von Neumann and Morgenstern:**
  DM is risk averse iff his marginal utility $U'$ is decreasing, more generally, if the utility function is strictly concave.

- **Theorem (Markowitz 1952):** Let the investor be rational and risk averse, let the utility function $U$ be quadratic. Then the contenders for optimality are the efficient portfolios of

$$\max \{ E[R(x,r)] \}$$

$$\min \{ V[R(x,r)] \} \quad \text{s.t. } x \in X$$
1.3 Risk Aversion and the E-V Problem

\[
\begin{align*}
\max & \ E \ [R(x,r)] \\
\min & \ V \ [R(x,r)] \quad \text{s.t. } x \in X
\end{align*}
\]
1.4 Multiple Criteria

- Standard investor maximizes return (price appreciation). Nonstandard investors might care about...
  
  (S) max \{ return \}  
  (S) max \{ relative return \}  \text{(against benchmark)}  
  (D) min \{ modified duration \}  \text{(neg. interest rate sensitivity)}  
  (D) min \{ turnover ratio \}  \text{(% of portfolio sold this period)}  

... i.e., the Utility Function is multivariate \( U(R,z_2,\ldots,z_k) \)

- Focus on two cases:
  (i) return and one additional deterministic objective  
  (ii) return and one additional stochastic objective
1.4 Multiple Criteria: Deterministic Objective

- **Theorem:** Consider a rational investor exhibiting risk aversion towards the stochastic objective (return) and continuous utility for the deterministic utility. Then the contenders for expected utility maximization are equal to the efficient portfolios of the $E-V-z_2$ problem.

- **E-V-M Problem:** Let $z_2$ denote modified duration.

\[
\begin{align*}
\text{(S)} & \quad \text{max \{ return \}} \\
\text{(D)} & \quad \text{min \{ modified duration \}} \\
\leftrightarrow & \\
\text{(D)} & \quad \text{max \{ expected return \}} \\
\text{(D)} & \quad \text{min \{ variance \}} \\
\text{(D)} & \quad \text{min \{ modified duration \}}
\end{align*}
\]
1.4 Multiple Criteria: Stochastic Objective

- **Theorem:** Consider a rational investor with a strictly concave quadratic bivariate utility function. Then the contenders for expected utility maximization are the efficient portfolios for the $E-V-E-V-Cov$ problem.

- **E-V-T Problem:** Let $z_2$ denote relative return.

\[
\begin{align*}
\text{max} \{ \text{return} \} & \quad (S) \\
\text{max} \{ \text{relative return} \} & \quad (S)
\end{align*}
\]

\[
\begin{align*}
(D) & \quad \text{max} \{ \text{expected return} \} \\
(D) & \quad \text{min} \{ \text{variance} \} \\
(D) & \quad \text{max} \{ \text{relative return} \} \\
(D) & \quad \text{min} \{ \text{relative variance} \} \\
(D) & \quad \text{min} \{ \text{covariance of } z_1 \text{ and } z_2 \}
\end{align*}
\]

- **Simplification:** 3rd and 5th objective can be omitted.
2. Computation of efficient portfolios for E-V-M

2.1 Quadratic single-parametric programming
2.2 Quadratic bi-parametric programming
2.3 Computational experience
2.1 Quadratic single-parametric programming

- Algorithms for the computation of the efficient frontier: critical line method (Markowitz) or three-phase method (Wolfe)

\[
\max \{ -V[R(x,r)] + \lambda E[R(x,r)] : x \in X \} \quad (\lambda > 0 \text{ parameter})
\]
2.1 Quadratic uni-parametric programming

- Efficient frontier consists of parabolic segments

parabolic segments
(contain fixed securities in varying proportions)
2.2 Quadratic bi-parametric programming

- Consider one stochastic criterion (portfolio return) and one deterministic linear criterion (modified duration).
  \[
  \max \{ -V[R(x,r)] , E[R(x,r)] , M(x) : x \in X \}
  \]

- Weighting approach according to the three-phase method
  \[
  \max \{ -V[R(x,r)] + \lambda_1 E[R(x,r)] + \lambda_2 M(x) : x \in X \}
  \]

again, fixed securities in varying proportions for each “stability set”
2.2 Quadratic bi-parametric programming

50 stocks with 482 platelets (S&P 1500 SuperComposite Index)

paraboloidal plates
(contain fixed securities in varying proportions)
2.2 Quadratic bi-parametric programming

500 stocks with 4625 platelets (S&P 1500 SuperComposite Index)
2.3 Computational testing

- Computation times and number of platelets for $n$ stocks (S&P SuperComposite 1500)

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- Generally: Time increase for additional objective is moderate
3. Nonlinear Criteria

3.1 Turnover Ratio (E-V-R)
3.2 Tracking Error (E-V-T)
3.1 Turnover Ratio

- Let $c$ denote the investment proportions of the current portfolio. Turnover Ratio is a **nonlinear objective**:

$$
\min \{ \sum_i \max (x_i - c_i, 0) : x \in X \}
$$

- Since the objective is separable, it may be linearized by introducing $n$ auxiliary variables and $n$ constraints. This causes extremely long running times for a large market (e.g. $n = 1000$). A specialized algorithm is able to solve this problem in almost the same time as the linear $E-V-M$ problem.
3.2 Tracking Error

Let $w$ denote the investment proportions of the benchmark, then

$$relV \left[ R(x, r) \right] = V \left[ R(x, r) \right] - V \left[ R(w, r) \right].$$

The objective is quadratic and satisfies the following equation

$$relV \left[ R(x, r) \right] = V \left[ R(x, r) \right] - 2 Cov \left[ R(x, r), R(w, r) \right] + V \left[ R(w, r) \right].$$

When $w$ is fixed, the covariance term is linear in $x$, so we set

$$2 Cov \left[ R(x, r), R(w, r) \right] = \gamma^T x.$$

Inserting this into the weighted-sum problem, we obtain

$$\max \left\{ -V \left[ R(x, r) \right] + \lambda_1 E \left[ R(x, r) \right] + \lambda_2 \gamma^T x : x \in X \right\}$$

where $0 \leq \lambda_2 \leq 1$. An algorithm keeping an upper bound on $\lambda_2$ solves the **E-V-T** (Tracking Error) Problem.