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OPTION IMPLIED TREES IN THE PRESENCE OF DIFFERENT IMPLIED VOLATILITIES

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Muzzioli S., Torricelli C. "Implied trees in illiquid markets, a Choquet pricing approach" International Journal of Intelligent Systems, 17, (6), 577-594, 2002.

Moriggia V., Muzzioli S., Torricelli C., "Option implied trees when the put call parity is not fulfilled", ***Materiali di discussione n.448, Novembre 2003, Dipartimento di Economia Politica, Università degli studi di Modena e Reggio Emilia.***

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Motivation

- Need to price exotic options consistently to Standard European options

- Find a tree for the underlying that reflects the information contained in option prices.

- Call prices and put prices may carry different information on the underlying.

Empirical evidence:

- 1) **Smile effect** (volatility varies with moneyness and time to expiration)

Rubinstein (The Journal of Finance, 1994)

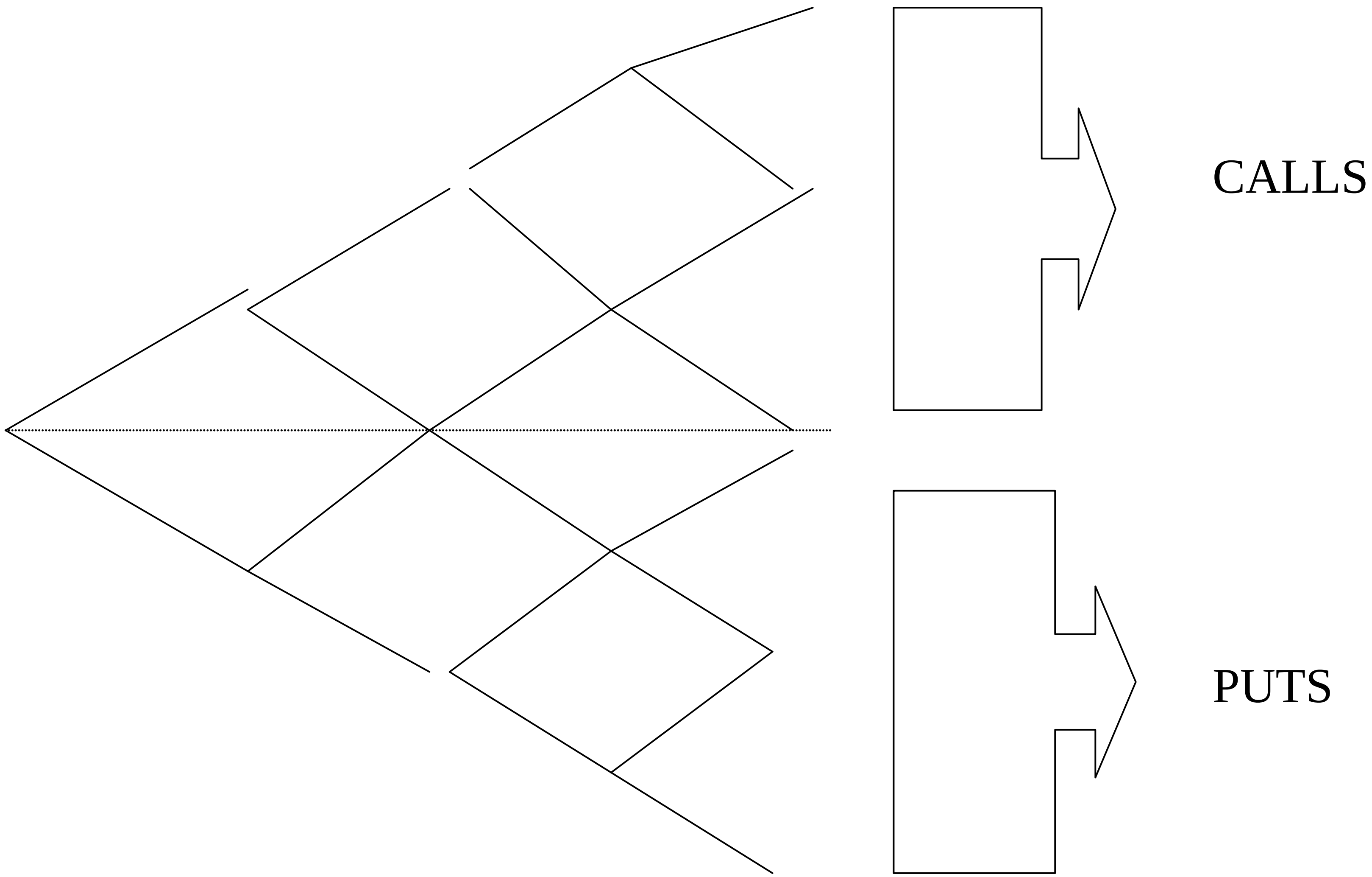
Derman & Kani (Risk, 1994)

- 2) volatility implied by **call** prices is different from the one implied by **put** prices

Kamara and Miller (1995), Chesney, Gibson and Loubergé (1995), Pena et al. (1999), Mittnick and Rieken (2000), Cavallo and Mammola (2000) Capelle et al. (2001), Brunetti and Torricelli (2003).

Smile call and Smile put

THE DERMAN AND KANI MODEL:



- **Prices are observed with measurement errors:** small errors in any of the input may produce large errors in the implied volatility, especially for far from the money options.

  Quoting Hentshle (2003): "*Unfortunately many authors preclude the cancellation of errors across puts and calls by using only the more liquid out of the money options. Unless underlying asset prices and dividend rates are observed with high precision, this practice can result in a substantial loss of efficiency*"

MT methodology

- a) construct two implied trees, one using only the information given by call options and the other using only information given by put options;
- b) aggregate the two trees: no arbitrage check
- c) take the implied stock prices as bounds for an interval of prices.
- d) take the implied probabilities as bounds for an interval of probabilities

Aim of this work

- Different methodology (w.r.t. Muzzioli and Torricelli (2002)) in order to imply the interval of artificial probabilities at each node of the tree.
- Empirical validation of the implied tree obtained, both in the sample and out of sample.
- Comparison with Derman and Kani's.

Plan of the presentation

- Muzzioli and Torricelli 2002 model
- The derivation of the r.n.p. interval
- The calibration procedure

Inputs of the example

- Date: 6th of January 2000
- Maturity: February
- The term structure of risk-less interest rates: interpolated risk-less rate is 3,19%
- the stock price at time zero = 6949,47
- the interpolated smile function for

  Calls: σC = 0,38132547 - 0,00001995X

  Puts: σP = 0,44034707 - 0,00001964X
- N. of levels: 6

The call implied tree

					8073,957
				7490,493	
			7248,876		7249,436
		6995,608		6995,974	
	6750,36		6750,561		6750,764
6494,47		6494,47		6494,47	
	6248,28		6248,094		6247,907
		5992,192		5991,787	
			5747,59		5747,01
				5494,278	
					5254,56

The put implied tree

					8073,957
				7754,148	
			7444,882		7445,777
		7122,997		7123,569	
	6813,899		6814,208		6814,519
6494,47		6494,47		6494,47	
	6190,015		6189,735		6189,452
		5876,976		5876,378	
			5581,05		5580,217
				5278,776	
					4995,976

The Pc-implied tree

					8073,96
					8073,96
				7754,15	
				7490,49	
			7444,88		7445,78
			7248,88		7249,44
		7123		7123,57	
		6995,61		6995,97	
	6813,9		6814,21		6814,52
	6750,36		6750,56		6750,76
6494,47		6494,47		6494,47	
6494,47		6494,47		6494,47	
	6248,28		6248,09		6247,91
	6190,02		6189,73		6189,45
		5992,19		5991,79	
		5876,98		5876,38	
			5747,59		5747,01
			5581,05		5580,22
				5494,28	
				5278,78	
					5254,56
					4995,98

Aggregating the two trees:

By no arbitrage:

$$\overline{S}(i-1, j)*(1+r) < \underline{S}(i, j+1) < \underline{S}(i, j)*(1+r)$$

$$\overline{S}(i-1, j)*(1+r) < \overline{S}(i, j+1) < \underline{S}(i, j)*(1+r)$$

$[\underline{S}(i, j), \overline{S}(i, j)]$

$[\underline{S}(i-1,j) \ , \ \overline{S}(i-1,j)]$

$\rightarrow [\underline{S}(i, j+1), \overline{S}(i, j+1)]$

Check: if only one of the two conditions is fulfilled, then take S(i,j+1) crisp and equal to the one that satisfies the no arbitrage condition,

if no one is fulfilled, then take S(i,j+1) crisp and equal to the following:

$$S(i,j+1) = \frac{(1+r)\left[\overline{S}(i-1, j) + \underline{S}(i, j)\right]}{2}$$

The risk neutral probabilities

- We drop assumption A3) of Muzzioli and Torricelli (2002) that suggests to imply the interval of artificial probabilities at each node, by taking the minimum and the maximum of the artificial probabilities implied by using only call or put options respectively.

- We derive endogenously the artificial probabilities by using the risk neutral valuation formula.

Derivation of the r.n.p.

$$\begin{cases} p\ (i+1, j+1) + (1 - p\ (i+1, j+1)) = 1 \\ [\underline{S}(i, j), \overline{S}(i, j)]e^{r\Delta t} = p(i+1, j+1)[\underline{S}(i+1, j+1), \overline{S}(i+1, j+1)] \\ \qquad\qquad\qquad + (1 - p(i+1, j+1))[\underline{S}(i, j+1), \overline{S}(i, j+1)] \end{cases}$$

$$p\ (i+1, j+1) = \frac{[\underline{S}(i, j), \overline{S}(i, j)]e^{r\Delta t} - [\underline{S}(i, j+1), \overline{S}(i, j+1)]}{[\underline{S}(i+1, j+1), \overline{S}(i+1, j+1)] - [\underline{S}(i, j+1), \overline{S}(i, j+1)]}$$

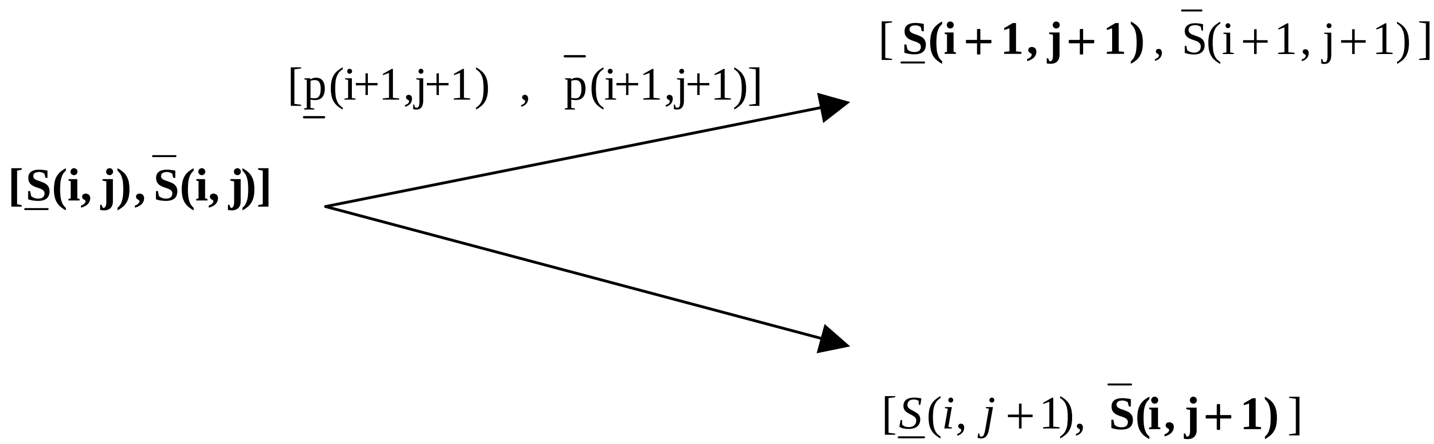
The r.n.p.

By noting that: p(i+1, j+1) is increasing in S(i,j) and decreasing in S(i,j+1) and S(i+1,j+1), we get:

$$[\underline{p}(i+1, j+1), \overline{p}(i+1, j+1)] =$$

$$\left[\frac{\underline{S}(i,j)e^{r\Delta t} - \overline{S}(i,j+1)}{\overline{S}(i+1,j+1) - \overline{S}(i,j+1)}, \frac{\overline{S}(i,j)e^{r\Delta t} - \underline{S}(i,j+1)}{\underline{S}(i+1,j+1) - \underline{S}(i,j+1)} \right]$$

THE ONE PERIOD MODEL:



The tree calibration:

- In order to analyse the performance of the model both in the sample and out of sample we need to <u>compare model prices with market prices</u>, i.e. we need to extract one single price from the interval.

- We are interested in determining the underlying price process that better fits the market prices of the options.

- Two different methods: one or two parameters

One parameter

Define:

$$S_{\alpha}(i,j) = \alpha \underline{S}(i,j) + (1-\alpha)\overline{S}(i,j)$$

where $\alpha \in [0,1]$.

We estimate the parameter α, by solving the following non linear optimisation problem:

$$\min_{\alpha} \sum_{i=1}^{m} (f_T(\alpha) - f_M)^2$$

s.t. $\alpha \in [0,1]$

Two parameters: α and β

- In order to separate the effect of the nodes of the upper part of the tree from the lower part we use two different parameters

- To parametrise the stock price we use:

  α for the upper part

  β for the lower

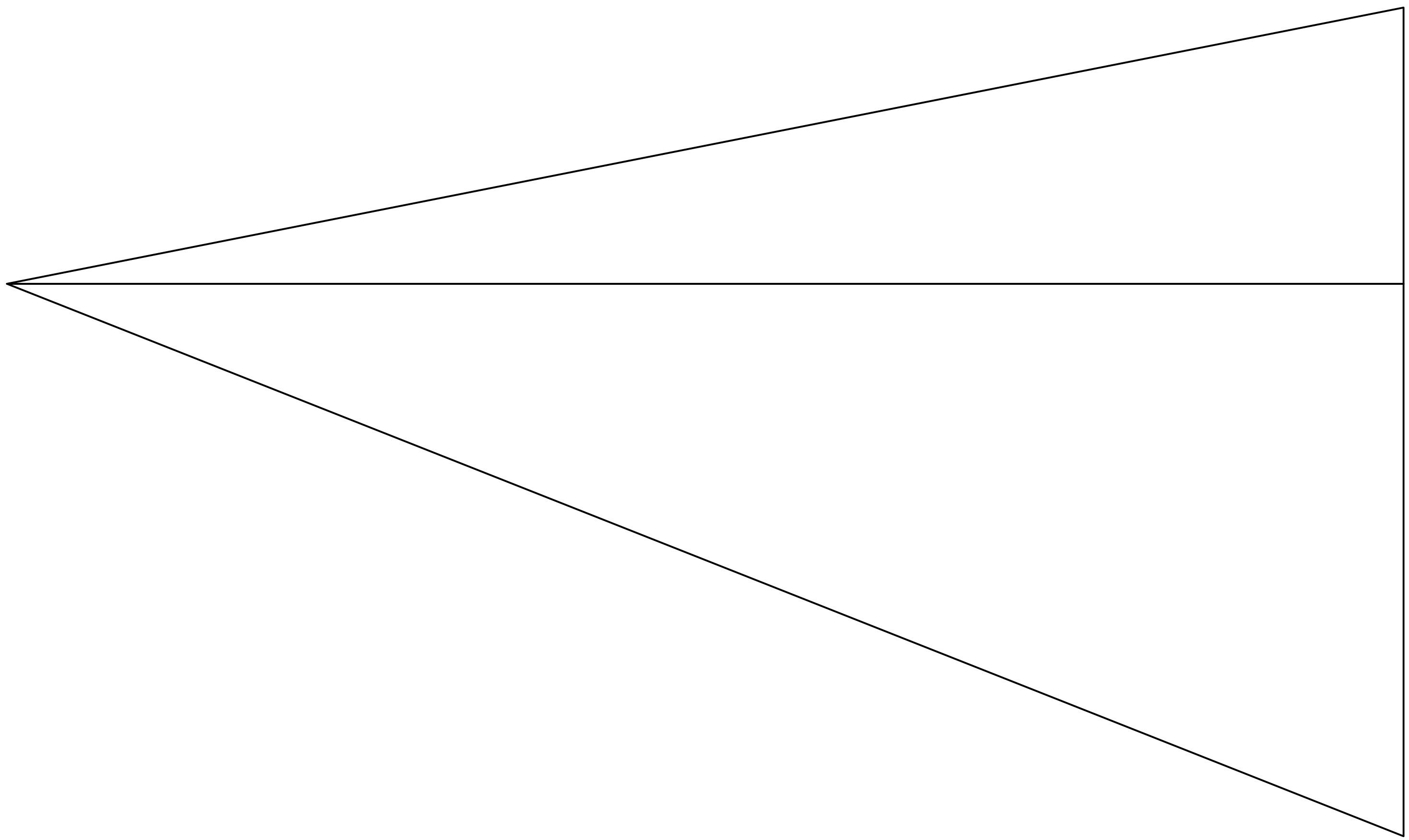
- We solve the non linear optimisation problem

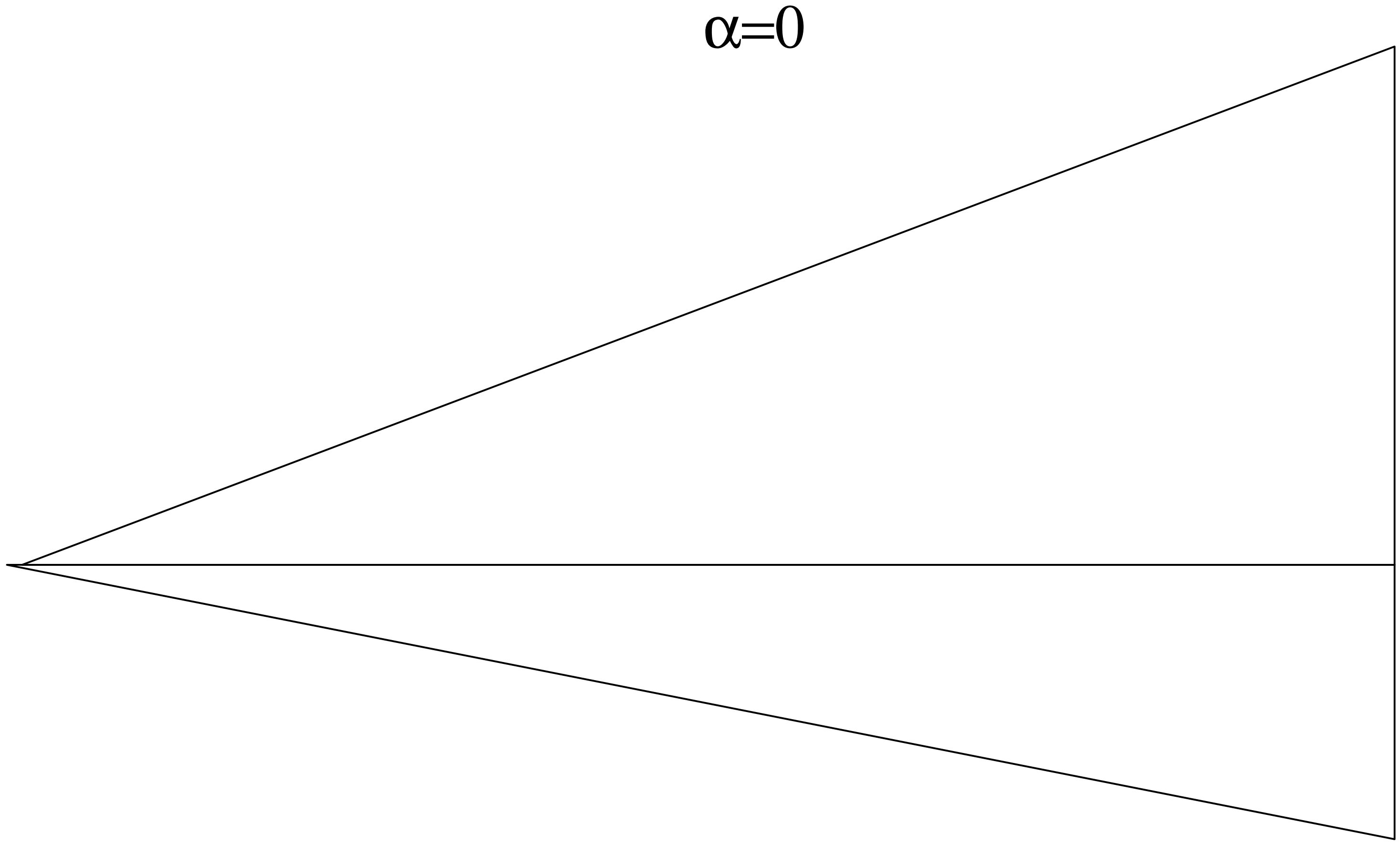
$$\min_{\alpha,\beta} \sum_{i=1}^{m} (f_T(\alpha, \beta) - f_M)^2$$

s.t. $\alpha \in [0,1]$ $\beta \in [0,1]$

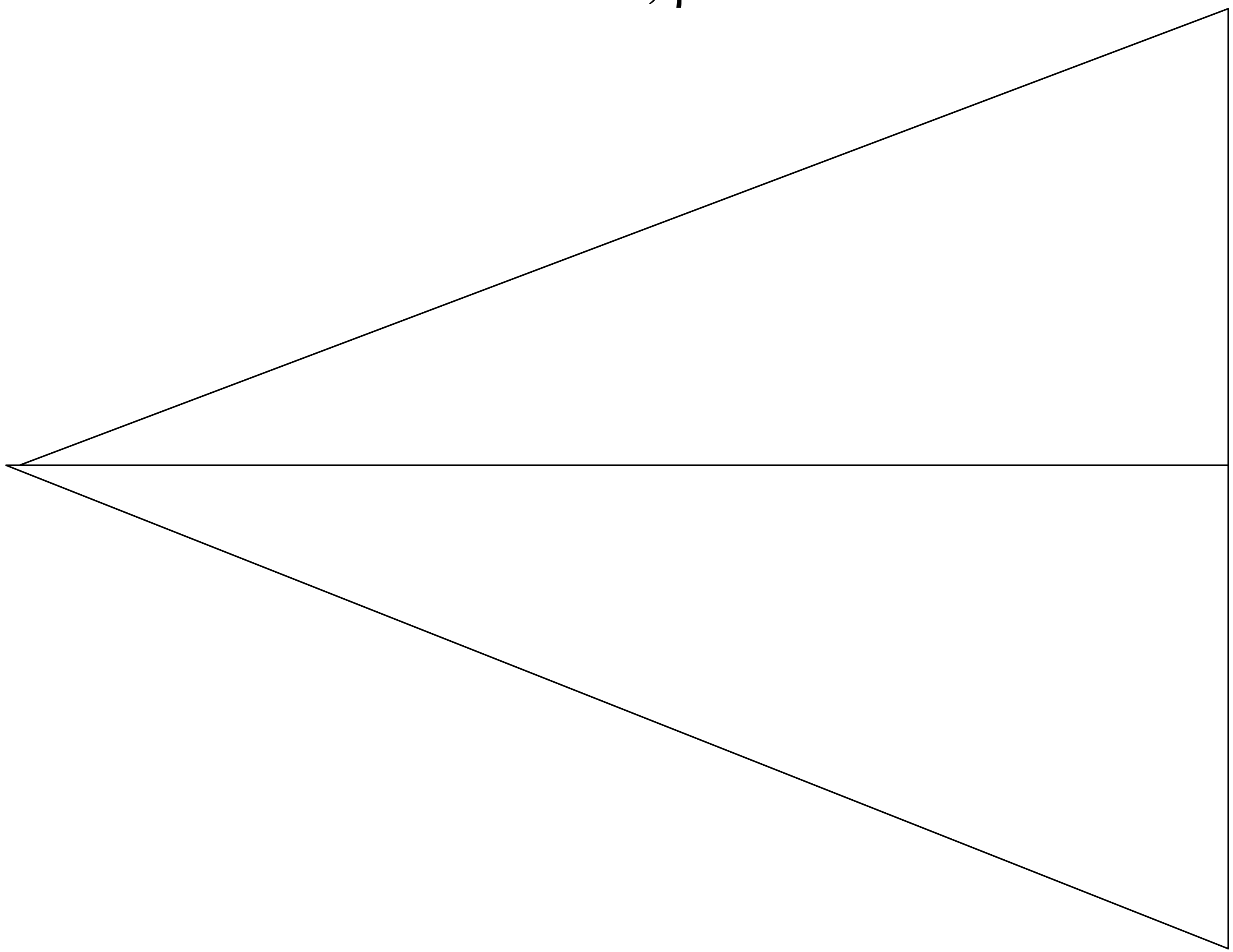
$$\alpha = 1$$

$$S_\alpha(i, j) = \alpha \underline{S}(i, j) + (1-\alpha)\overline{S}(i, j)$$

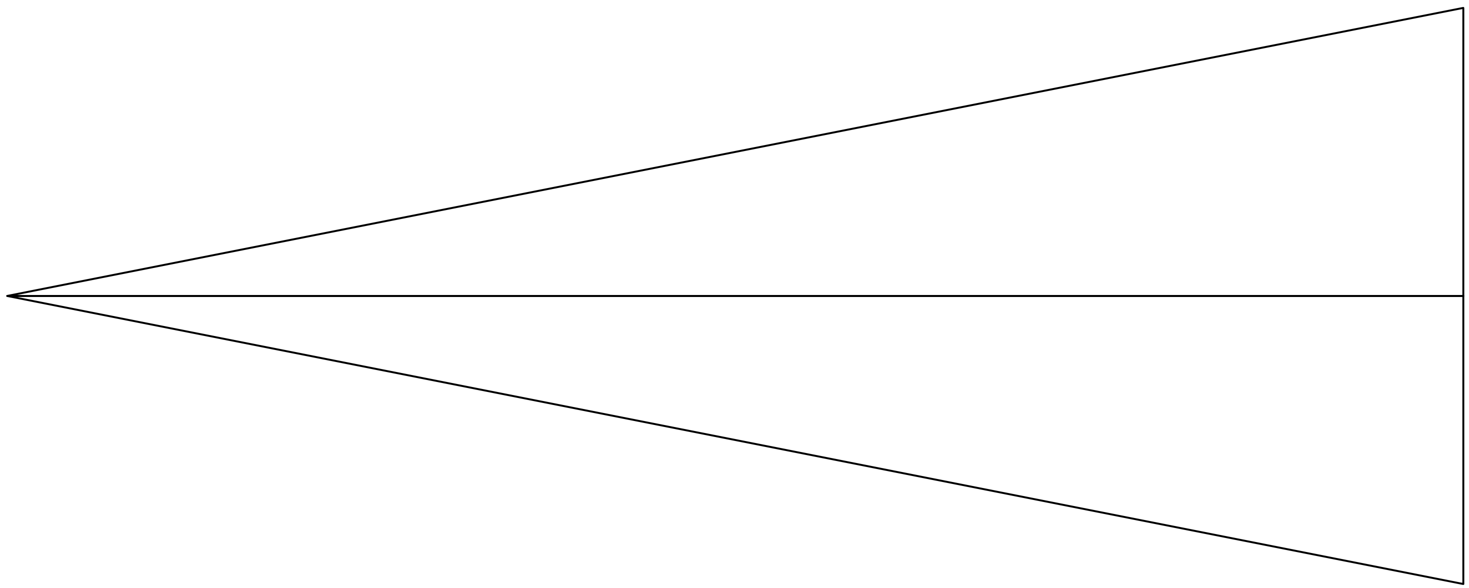



α=0

$\alpha=0$, $\beta=1$



$\alpha=1$, $\beta=0$



Why DAX-index options?

The options are European (NO EARLY EXERCISE)

DAX index is adjusted for dividends, stocks splits and changes in capital: since dividends are assumed to be reinvested into the shares, they do not affect the index value (NO DIVIDEND ADJUSTMENTS)

The Data Set

- ***Date***: from 04/01/99 to 30/12/00
- ***Underlying***: Dax index (last)
- ***Options***: Call & Put options on the Dax (last)
- ***Risk Free***: Fibor rate (linearly interpolated)
- ***Maturity:*** up to 12 months
- ***Smile***: interpolated by a linear function (y=a-bx)
- **n:** 25 and 26 levels

- 2882 date –maturity classes. (average of 76 options)

Derman and Kani

- Inputs: the underlying spot price, the risk-less interest rate and the smile.

- smile of call options for strikes greater or equal to the value of the underlying and the smile of put options for strikes less than the value of the underlying.

- no-arbitrage check, the Barle and Cakici condition is used.

MT tree

- Inputs: the underlying value, the risk-less interest rate and the two smile functions for call and put options.
- We derive two trees as in (DK), one using the smile of call options and the other using the smile of put options,
- Aggregate them in order to have a unique implied tree with interval values for the stock prices.
- no arbitrage check.
- We run the non linear optimisation routine in order to get crisp values for the stock prices and probabilities.

In the sample performance

Indicators:

$$SSE = \frac{1}{m}\sum_{i=1}^{m}(P_i^T - P_i^M)^2$$

$$MISP = \frac{\sum_{i=1}^{m}\left(\frac{P_i^T - P_i^M}{P_i^M}\right)}{\sum_{i=1}^{m}\left|\frac{P_i^T - P_i^M}{P_i^M}\right|}$$

In the sample performance

	SSE	MISP	SSE Call	MISP Call	SSE Put	MISP Put
DK	1947.77	-0.61	1618.53	-0.46	2277.00	-0.64
Method 1	248.13	-0.12	250.70	0.16	245.55	-0.35
Method 2	230.67	-0.32	197.40	-0.01	263.94	-0.47

The average parameter estimates:

Method 1: alpha = 0,65

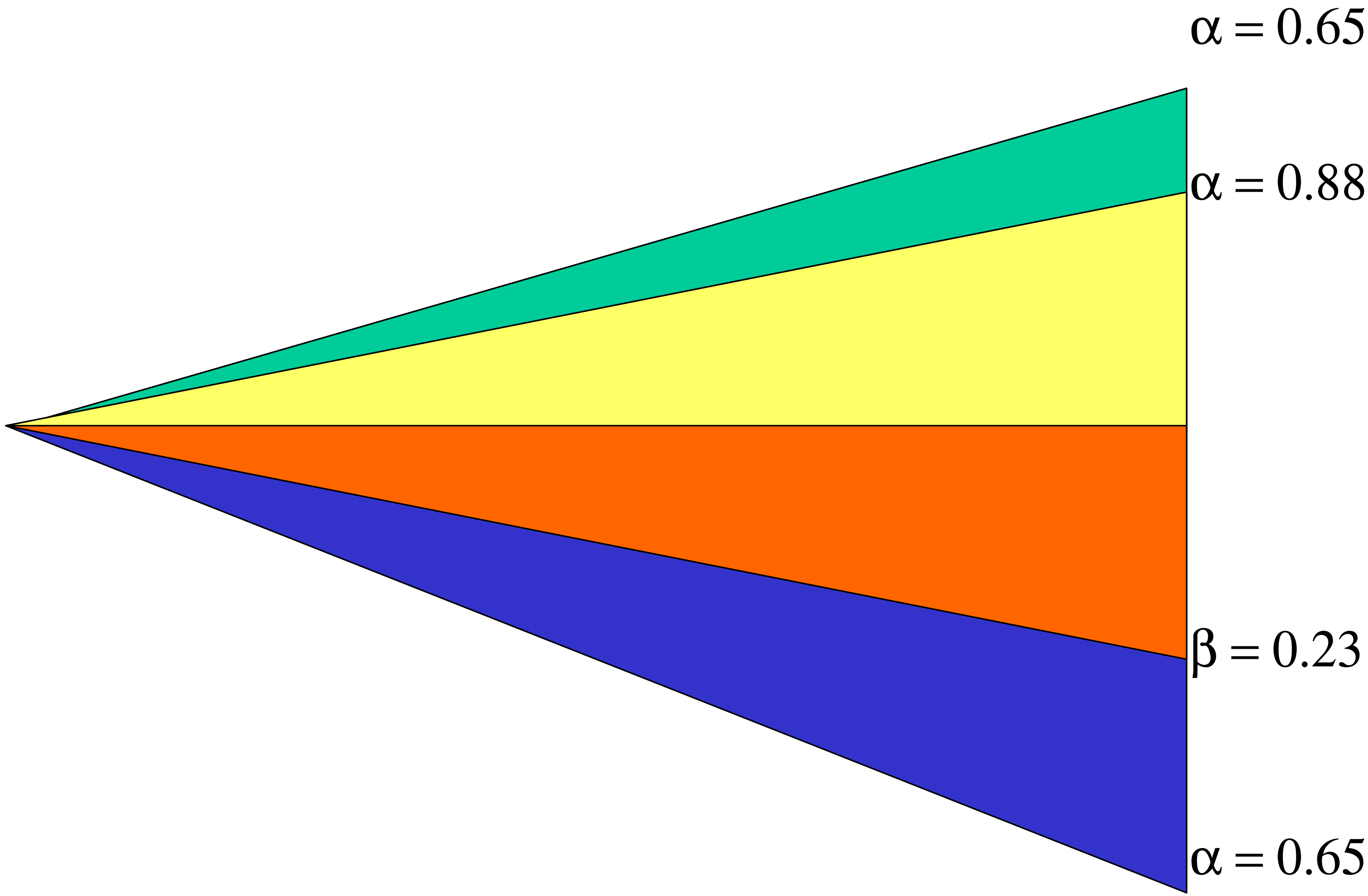
Method 2: alpha = 0,88 and beta = 0,23.

In the sample performance

- Models 1 and 2 are better than DK.

- Slightly better fit of Method 2, can thus be explained by the difference in the two parameter estimates and the associated lower standard errors.

- Method 2 has a lower volatility than method 1 (the underpricing of each option class is bigger than in Method 1).


α = 0.65
α = 0.88
β = 0.23
α = 0.65

TTM classes

DK						
	SSE	**MIS**	**SSE Call**	**MISP Call**	**SSE Put**	**MISP Put**
0-2 month	912.3868	-0.5452	844.3079	-0.5680	980.4657	-0.5438
2-4 month	1430.3877	-0.5917	1253.4572	-0.4489	1607.3183	-0.6134
4-8 month	2500.6870	-0.6943	2066.7509	-0.4331	2934.6232	-0.7500
8-12 month	3435.8224	-0.6249	2672.3172	-0.3365	4199.3275	-0.6990
Method 1						
	SSE	**MIS**	**SSE Call**	**MISP Call**	**SSE Put**	**MISP Put**
0-2 month	229.7189	0.0758	248.6046	0.0044	210.8332	-0.0795
2-4 month	216.8319	-0.1321	221.8131	0.1075	211.8507	-0.3474
4-8 month	273.0739	-0.2918	284.0246	0.1956	262.1233	-0.5435
8-12 month	280.5897	-0.2101	247.2621	0.3912	313.9173	-0.5664
Method 2						
	SSE	**MIS**	**SSE Call**	**MISP Call**	**SSE Put**	**MISP Put**
0-2 month	189.4195	-0.1173	174.8312	-0.1264	204.0077	-0.2423
2-4 month	215.6399	-0.3613	172.1900	-0.0870	259.0898	-0.4311
4-8 month	256.3992	-0.4466	234.2797	0.0604	278.5188	-0.6370
8-12 month	280.4152	-0.4356	217.3034	0.1880	343.5269	-0.6578

TTM classes

- The underpricing of the Derman and Kani method is severe, particularly for puts.
- Methods 1 and 2 are much better than Derman and Kani's, especially for **long term options**.
- Overall Method 2 obtains a better performance than Method 1 in terms of SSE, while it underprices more (overprices less) than Method 1.

Parameter estimates:

	Method 1 Alpha	**Method2 alpha**	**Method2 beta**
0-2 month	0,76	0,87	0,14
	(0,41)	(0,27)	(0,29)
2-4 month	0,59	0,85	0,18
	(0,44)	(0,28)	(0,30)
4-8 month	0,66	0,88	0,32
	(0,40)	(0,24)	(0,35)
8-12 month	0,53	0,89	0,30
	(0,39)	(0,18)	(0,35)

The moneyness classes

- the indicator of moneyness: $M = S/(Ke^{-rT})$
- **DOM** (call options: $M < 0{,}9$ and put options: $M \geq 1{,}1$),
- **OM** (out-of-the-money, call options: $0{,}9 \leq M < 0{,}98$ and put options: $1{,}02 < M < 1{,}1$),
- **AM** (at-the-money, call and put options: $0{,}98 \leq M \leq 1{,}02$),
- **IM** (in-the-money, call options: $1{,}02 < M < 1{,}1$ and put options $0{,}9 \leq M < 0{,}98$),
- **DIM** (deep-in-the-money, call options: $M \geq 1{,}1$ and put options: $M < 0{,}9$).

The moneyness classes

DK						
	SSE	**MIS**	**SSE Call**	**MISP Call**	**SSE Put**	**MISP Put**
DOM	1922.343	-0.70966	177.1344	-0.60033	3667.551	-0.82091
OM	2274.105	-0.20825	528.8513	-0.02289	4019.359	-0.37643
AM	1812.499	-0.10748	1440.463	-0.01663	2184.536	-0.36978
IM	2038.402	-0.27001	3081.275	-0.19746	995.5284	-0.39543
DIM	1785.096	-0.38826	2945.494	-0.35411	624.6985	-0.46122
Method 1						
	SSE	**MIS**	**SSE Call**	**MISP Call**	**SSE Put**	**MISP Put**
DOM	179.9587	-0.27457	154.5186	-0.18519	205.3988	-0.51937
OM	321.09	0.375748	365.0524	0.424837	277.1277	0.074633
AM	275.9532	0.48038	309.0861	0.662258	242.8203	0.270609
IM	238.7734	0.365786	251.0423	0.662797	226.5046	0.024515
DIM	260.8418	-0.10528	224.1132	0.196358	297.5705	-0.36111
Method 2						
	SSE	**MIS**	**SSE Call**	**MISP Call**	**SSE Put**	**MISP Put**
DOM	187.8214	-0.35151	146.9532	-0.24941	228.6895	-0.60818
OM	281.6242	0.513863	271.4583	0.282644	291.7902	-0.14206
AM	228.3327	0.199579	193.673	0.448042	262.9924	-0.05861
IM	193.5049	0.049939	169.1523	0.305628	217.8575	-0.16525
DIM	265.5256	-0.24582	218.9244	-0.0448	312.1268	-0.39966

The moneyness classes

- The SSE indicates that Methods 1 and 2 perform better than Derman and Kani in each class of moneyness, particularly for in-the-money calls and out-of-the-money puts.
- The worst fit of the DK method obtains in the lower part of the tree, i.e. the one derived by using out of the money puts.
- The MISP indicates that the Derman and Kani method underprices most classes of options: the highest underpricing obtains for DIM and DOM call and put options.
- As expected, Method 2 underprices every option class more than Method 1. Method 2 obtains a better pricing performance for IM AM and OM options, the opposite holds for DIM and DOM options.

Out of sample performance

- Do more parameters obtain a better fit?
- Use date t estimated parameters (smile of calls and puts; α; α and β) to compute date t+1 theoretical prices (S, ttm, r)
- Compare with market prices.

Out of sample performance:

	SSE	MISP	SSE Call	MISP Call	SSE Put	MISP Put
DK	2231.897	-0.67734	1869.512	-0.46907	2594.282	-0.71731
Method 1	369.1174	-0.12334	382.2205	0.094889	356.0143	-0.38221
Method 2	351.5752	-0.28665	345.8505	-0.04026	357.3	-0.54025

In the sample:

	SSE	MISP	SSE Call	MISP Call	SSE Put	MISP Put
DK	1947.77	-0.61	1618.53	-0.46	2277.00	-0.64
Method 1	248.13	-0.12	250.70	0.16	245.55	-0.35
Method 2	230.67	-0.32	197.40	-0.01	263.94	-0.47

TTM classes

DK						
	SSE	**MIS**	**SSE Call**	**MISP Call**	**SSE Put**	**MISP Put**
0-2 month	965.6042	-0.69642	940.954	-0.56826	990.2543	-0.77649
2-4 month	1691.243	-0.63073	1589.862	-0.43734	1792.625	-0.72412
4-8 month	2903.526	-0.66264	2560.587	-0.43226	3246.465	-0.76295
8-12 month	4337.149	-0.60703	3871.623	-0.30816	4802.675	-0.7581
Method 1						
	SSE	**MIS**	**SSE Call**	**MISP Call**	**SSE Put**	**MISP Put**
0-2 month	329.1239	0.006393	339.1208	-0.04593	319.1271	-0.17244
2-4 month	332.6885	-0.09624	343.828	0.062299	321.549	-0.35708
4-8 month	399.3219	-0.25671	412.7486	0.129249	385.8951	-0.52808
8-12 month	434.2526	-0.20568	453.7423	0.306936	414.7629	-0.57229
Method 2						
	SSE	**MIS**	**SSE Call**	**MISP Call**	**SSE Put**	**MISP Put**
0-2 month	289.3947	-0.16113	294.5701	-0.15243	284.2192	-0.36957
2-4 month	321.7942	-0.27962	312.7215	-0.06977	330.867	-0.5146
4-8 month	385.8568	-0.38599	372.0952	0.012706	399.6184	-0.6614
8-12 month	441.7773	-0.38359	430.9483	0.10778	452.6063	-0.70304

TTM classes

- The underpricing of the Derman and Kani method is severe, particularly for puts.
- Methods 1 and 2 are much better than Derman and Kani's, especially for long term options.
- Method 2 obtains a better performance than Method 1 in terms of SSE, while it underprices more (overprices less) than Method 1.

The moneyness classes

DK						
	SSE	**MIS**	**SSE Call**	**MISP Call**	**SSE Put**	**MISP Put**
DOM	1872.58	-0.6994	190.278	-0.5391	3554.883	-0.8176
OM	2224.094	-0.33895	607.766	-0.11783	3840.422	-0.50886
AM	1844.872	-0.29154	1663.136	-0.14774	2026.607	-0.44633
IM	2189.924	-0.28278	3375.471	-0.22743	1004.378	-0.38233
DIM	1988.69	-0.31682	3244.629	-0.28341	732.7516	-0.38793
Method 1						
	SSE	**MIS**	**SSE Call**	**MISP Call**	**SSE Put**	**MISP Put**
DOM	162.9669	-0.29407	141.126	-0.19327	184.8078	-0.55881
OM	296.8426	0.281002	349.5409	0.374597	244.1443	0.058412
AM	346.9351	0.380268	395.2703	0.547142	298.5998	0.170706
IM	481.0347	0.237822	484.915	0.445035	477.1544	0.013654
DIM	534.0856	-0.05929	497.3995	0.116536	570.7717	-0.23939
Method 2						
	SSE	**MIS**	**SSE Call**	**MISP Call**	**SSE Put**	**MISP Put**
DOM	168.6209	-0.36931	132.7025	-0.24457	204.5394	-0.66367
OM	260.8862	0.112091	269.7659	0.244414	252.0066	-0.21347
AM	302.5933	0.113091	314.0793	0.309581	291.1072	-0.13808
IM	433.0772	0.051547	423.3172	0.241822	442.8373	-0.14074
DIM	532.8754	-0.13305	487.8545	0.013539	577.8962	-0.27858

The moneyness classes

- The SSE indicates that Methods 1 and 2 perform better than Derman and Kani in each class of moneyness, particularly for in-the-money calls and out-of-the-money puts.
- The worst fit of the DK method obtains in the lower part of the tree, i.e. the one derived by using out of the money puts.
- The MISP indicates that the Derman and Kani method underprices most classes of options: the highest underpricing obtains for DOM call and put options.
- As expected, Method 2 underprices every option class more than Method 1. Method 2 obtains a better pricing performance for IM AM options, the opposite holds for DIM OM and DOM options.

The performance for n =50 and 51

	SSE	MISP	SSE Call	MISP Call	SSE Put	MISP Put
DK	3433,344	-0,77354	3179,245	-0,61759	3687,442	-0,84928
Method 1	359,4763	-0,1559	377,7006	0,107417	341,252	-0,43878
Method 2	343,7804	-0,23541	352,8553	0,068775	334,7054	-0,53067

n=25 and 26

	SSE	MISP	SSE Call	MISP Call	SSE Put	MISP Put
DK	2231.897	-0.67734	1869.512	-0.46907	2594.282	-0.71731
Method 1	369.1174	-0.12334	382.2205	0.094889	356.0143	-0.38221
Method 2	351.5752	-0.28665	345.8505	-0.04026	357.3	-0.54025

Different objective function

$$\min_{v} \sum_{i=1}^{m} \left(\frac{f_T(v) - f_M}{f_M} \right)^2$$

	SSE	MISP	SSE Call	MISP Call	SSE Put	MISP Put
Method 1	373,9527	-0,08276	393,6426	0,144531	354,2629	-0,38126
Method 2	384,5589	-0,17903	385,4704	0,075156	383,6474	-0,46693

Min SSE:

	SSE	MISP	SSE Call	MISP Call	SSE Put	MISP Put
Method 1	369.1174	-0.12334	382.2205	0.094889	356.0143	-0.38221
Method 2	351.5752	-0.28665	345.8505	-0.04026	357.3	-0.54025

Comparison with MT 2002

<u>*ARPE=*</u> $\frac{1}{m}\sum_{i=1}^{m}\left(\frac{P_i^T - P_i^M}{P_i^M}\right)$

<u>*AAPE=*</u> $\frac{1}{m}\sum_{i=1}^{m}\left|P_i^T - P_i^M\right|$

	In the sample	In the sample	In the sample	Out of sample	Out of sample
	SSE	**ARPE**	**AAPE**	**ARPE**	**AAPE**
D&K	49525.56	-0.18163	14.022	-0.172	24.41
MT	7636.18	0.00366	6.1696	-0.0165	24.48
Method 1	3389.91	-0.05472	0.0903	-0.0507	0.09789
Method 2	2208.67	-0.06539	0.0879	-0.0642399	0.0994235

Conclusions and Research agenda

- The interval of prices for the derivative security better reflects the information available on the market both in the sample and out of sample.
- Better pricing of long term options and far from the money options.
- Method 2 is slightly better than Method 1 both in the sample and out of sample.
- Future research:
  - different parameterisation of the stock prices
  - ……