Utilising basket products to hedge credit risk

Luca Taschini
University of Bergamo
Via dei Caniana, 2
24127 Bergamo, Italy
Email: luca.taschini@unibg.it
What is Credit Risk?

Credit Risk can be seen as the possibility that an unexpected variation of the credit quality of a subject, with respect to which a credit exposure exists, creates a corresponding unexpected variation of the market value of the credit exposition.

There exist different typologies: *insolvency risk*; *migration risk*.

The credit risk components:

- **Expected loss**: it is the loss that a lender expects to achieve with respect to a given credit portfolio.

  \[ EL_{t1,t2} = p_{t1,t2} \cdot (1 - RR) \]

- **Unexpected loss**: it represents the measure of the variability of the loss rate with respect to the expected value.
Instruments to manage the risk

Currently the market of the derivatives offers numerous products to carry out hedging strategies calls Credit Derivatives. These are over-the-counter instruments that allow to:

- to separate;
- to confer a price;
- to transfer the credit risk implied in each credit-exposure.

Main contract typologies:

- against default
  - Credit default products*;
  - Basket products*;
- against downgrading risk
  - Credit spread products;
  - Total rate of return swaps.

* these are instruments that provide insurance against a particular company (for the CDS) or against two or more companies (for the basket) defaulting on its debt. The company is known as the reference entity and a default by the company is known as a credit event. The buyer of the protection (the protection buyer) makes periodic payments to the seller of protection (the protection seller) at a predetermined fixed rate per year. The payments continue until the end of the life of the contract or until a credit event, whichever is earlier.
Evaluation of CDS using Hull & White model

Now we can evaluate a CDS contract looking at the future outcomes and incomes of the protection buyer:

- **Protection cost**
  \[ \omega \cdot \sum_{i=0}^{T} p_i \cdot u(i) + \omega \cdot \pi \cdot u(T) \]
  we obtain \( w \) such that the two values will coincide

- **Value of the protection**
  \[ \sum_{i=0}^{T} (1 - \hat{R}) \cdot p_i \cdot v(i) \]

i.e.

\[
\omega = \frac{\sum_{i=0}^{T} (1 - \hat{R}) \cdot p_i \cdot v(i)}{\sum_{i=0}^{T} p_i \cdot u(i) + \pi \cdot u(T)}
\]
Hull & White model: remarks

Spread

<table>
<thead>
<tr>
<th>Spread bps</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,900</td>
</tr>
<tr>
<td>2,400</td>
</tr>
<tr>
<td>2,900</td>
</tr>
<tr>
<td>3,400</td>
</tr>
<tr>
<td>3,900</td>
</tr>
<tr>
<td>4,400</td>
</tr>
<tr>
<td>4,900</td>
</tr>
</tbody>
</table>

Recovery Rate

0 0,3 0,5 0,75 0,9 0,95
Hull & White model: implementation

- Reference entities in euro-area issuer of corporate bonds with rating A1:
  - Carrefour S.A.
  - Enel SpA
  - Unilever NV
  - Dresdner Bank A.G.
  - Erste Bank
  - Volkswagen Finance

Risk-neutral default probabilities computing using market prices data of bonds based on Hull & White approach.
Hull & White model: implementation

Let assume to have a portfolio in which there is one bonds of each reference entities considered and let assume we want to carry out an hedging strategy using credit default swaps with an horizon time of 5 years which will protect the total investment from defaults: in this case we will have a total cost of 3,729,780 euro.

<table>
<thead>
<tr>
<th>Reference Entities</th>
<th>Spread in bps</th>
<th>Spread in euro (1,000)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Carrefour S.A.</td>
<td>72,06928969</td>
<td>720,693</td>
</tr>
<tr>
<td>Dresdner Bank A.G.</td>
<td>86,20980168</td>
<td>862,098</td>
</tr>
<tr>
<td>Enel SpA</td>
<td>81,02908642</td>
<td>810,291</td>
</tr>
<tr>
<td>Erste Bank</td>
<td>77,8027794</td>
<td>778,028</td>
</tr>
<tr>
<td>Unilever NV</td>
<td>74,46061246</td>
<td>744,606</td>
</tr>
<tr>
<td>Volkswagen Finance</td>
<td>59,20932894</td>
<td>592,093</td>
</tr>
</tbody>
</table>

Table 1: Spreads of a CDS at 5 years.
Source: Our data.
Basket credit derivatives

Basket products are those financial contracts whose payout depends on the credit event characterising a portfolio of bonds over a determinated time horizon: thus their underling is the credit quality of more reference entities.

The most common basket is that one whose payout depend on the temporal ranking of the credit event (first-to-default, second-to-default).

Fundamental aspects:

- Default probability: in order to describe the survival time of each defaultable reference entity we introduce a variable called time-until-default and we construct its probability distribution.

- Joint default: once stated as compute the default probability of each reference entity we study how to compute the joint default probabilities.

Starting from the credit curves we can determine the marginal conditional default probabilities with reference to a stated time interval.
Characterize default using time-until-default

We introduce a random variable called *time-until-default* that represent the length of the survival time.

The probability distribution function of the survival time $T$ can be specified by the following distribution function

$$ F(t) = \text{Pr}(T \leq t) $$

which gives the probability that default occurs before $t$. And the corresponding probability density function:

$$ f(t) = \frac{dF(t)}{dt} $$

In studying survival data it is useful to define the *survival function* as:

$$ S(t) = 1 - F(t) = \text{Pr}(T > t) $$

that is the probability that the credit survives at time $t$. 
The time-until-default could be described also by the **hazard rate** function which gives the instantaneous default probability, i.e. the default probability of the credit over the time interval \([x, x+\Delta t]\) if it has survived to time \(x\):

\[
\Pr\{T \leq x + \Delta t \mid T > x\} = \frac{F(x + \Delta t) - F(x)}{1 - F(x)} \approx \frac{f(x) \cdot \Delta t}{1 - F(x)}
\]

where the **hazard rate** function is:

\[
h(x) = \frac{f(x)}{1 - F(x)}
\]

it gives the value of the conditional probability density of \(T\), i.e. the probability of default exactly at time \(x\) given the survival to that time.
Time-until-default & hazard rate

The survival function can be expressed in terms of the hazard rate as:

\[ S(t) = e^{-\int_{0}^{t} h(s)ds} \]

The probability density function of survival time of a credit can also be expressed using the hazard rate function:

\[ f(t) = h(t) \cdot S(t) = h(t) \cdot e^{-\int_{0}^{t} h(s)ds} \]

A typical assumption is that the hazard rate is a constant over certain period \([x, x+\Delta t]\), in this case the density function is:

\[ f(t) = h \cdot e^{-ht} \]
Basket products

The approach we used to pricing basket products is based on the Duffie’s model so-called “reduced” form. The idea behind this approach is that the credit event can be modelled as a Poisson process, with intensity rate \( h \) depending on the length of the time interval.

The general form of the Poisson distribution is:

\[
Poisson(x, h) = \frac{e^{-ht} \cdot (ht)^x}{x!}
\]

if we consider one company we are interested only in the case \( x = 0 \), i.e. when the company is not in default (in each other case, \( x > 0 \), the company is in default).

The probability that a company defaults before time \( t \) (the so-called "survival time") is measured by the time-until-default which function is Exponentially distributed:

\[F(t) = \Pr(T \leq t) = 1 - e^{-ht}\]
Pricing a first-to-default

The probability that each reference entity will survive until time $t$ is:

$$S(t) = \Pr(T > t) = e^{-h \cdot t}$$

with hazard rate $h$

Given $N$ reference entities, the multivariate survival joint-distribution is:

$$S(t) = e^{-H \cdot t}$$

with $H = \begin{cases} 
  h_1 + h_2 + \ldots + h_N & \text{independent issuer} \\
  h_1 + h_2 + \ldots + h_N + h_{1\ldots N} & \text{if a systematic risk factor exist} \\
\end{cases}$

or also $N \cdot h + h_{1\ldots N} = N \cdot \hat{h} - h_{1\ldots N}$

if all the issuers are equally risky
Incidence of the systematic factor

At total individual hazard rate parity \( (\hat{h}) \), the two components

\[ h \quad \rightarrow \quad \text{idiosyncratic risk} \]
\[ h_{1\ldots N} \quad \rightarrow \quad \text{systematic risk} \]

have different influence due to the increasing of the correlation between the lender.

In particular

\[ h = \frac{\hat{h} (1 - \rho)}{1 + \rho} \]

\[ h_{12\ldots N} = 2\hat{h} \frac{\rho}{1 + \rho} \]

In fact if

\[ \rho = 0 \quad \rightarrow \quad \begin{cases} h = \hat{h} \\ h_{1\ldots N} = 0 \end{cases} \]

\[ \rho = 1 \quad \rightarrow \quad \begin{cases} h = 0 \\ h_{1\ldots N} = \hat{h} \end{cases} \]
Pricing a first-to-default

The price of the first-to-default option is:

\[ P = L \cdot (1 - RR) \cdot \int_{0}^{T} e^{-rt} H e^{-H \cdot t} dt \]

nominal value  
loss given default  
total hazard rate of the portfolio

\[ = L \cdot (1 - RR) \cdot \frac{H}{r + H} \left(1 - e^{-T(r+H)}\right) \]

If we simulate different correlation values and keep constant \( \hat{h}_i \)

\[ \uparrow \rho \Rightarrow \uparrow h_{1\ldots N} \Rightarrow \hat{h}_i \text{Cost.} \Rightarrow \downarrow H \Rightarrow \downarrow P \]
Implementation of the model

For the implementation we have considered the same reference entities we used with the H&W model. Using the following risk-free rates and assuming a constant hazard rate function between each time interval (i.e. \( h(t) = h_i \) with \( t_{i-1} \leq t < t_i \)), we can compute the different hazard rates using the following expression:

\[
V(t_0) = \sum_{i=1}^{n} c_i \cdot e^{-\int_{t_0}^{t_i} [r(s)+(1-R(s))h(s)]ds}
\]

and assuming a constant hazard rate function between each time interval (i.e. \( h(t) = h_i \) with \( t_{i-1} \leq t < t_i \)).

Table: Risk-free rates from today up to 5 years.

<table>
<thead>
<tr>
<th>years</th>
<th>risk-free rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2,500%</td>
</tr>
<tr>
<td>1</td>
<td>2,239%</td>
</tr>
<tr>
<td>2</td>
<td>2,256%</td>
</tr>
<tr>
<td>3</td>
<td>2,539%</td>
</tr>
<tr>
<td>4</td>
<td>2,796%</td>
</tr>
<tr>
<td>5</td>
<td>3,012%</td>
</tr>
</tbody>
</table>

Source: Spot curve, Bloomberg.
Implementation of the model

✓ Using the pricing formula for a basket and considering the possible correlation values existing between the reference entities included in the basket (we assume a nominal invested capital $L$ equal to one and a recovery rate $RR = 39.46\%$) we get:

<table>
<thead>
<tr>
<th>Correlation</th>
<th>Premio in bps.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>888</td>
</tr>
<tr>
<td>0.1</td>
<td>768</td>
</tr>
<tr>
<td>0.2</td>
<td>665</td>
</tr>
<tr>
<td>0.3</td>
<td>577</td>
</tr>
<tr>
<td>0.4</td>
<td>500</td>
</tr>
<tr>
<td>0.5</td>
<td>433</td>
</tr>
<tr>
<td>0.6</td>
<td>373</td>
</tr>
<tr>
<td>0.7</td>
<td>320</td>
</tr>
<tr>
<td>0.8</td>
<td>272</td>
</tr>
<tr>
<td>0.9</td>
<td>228</td>
</tr>
<tr>
<td>0.99</td>
<td>193</td>
</tr>
<tr>
<td>1</td>
<td>189</td>
</tr>
</tbody>
</table>
Price of the first-to-default

Diversification reduces the risk
Not for a first-to-default
Comparing the two strategies: CDS

Applying the strategy based on credit default swap options we get the following results:

<table>
<thead>
<tr>
<th>Reference Entity</th>
<th>Spread in bps</th>
<th>Price in euro (1,000)</th>
</tr>
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<tbody>
<tr>
<td>Carrefour S.A.</td>
<td>72,06928869</td>
<td>720,693</td>
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<td>862,098</td>
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<td>744,606</td>
</tr>
<tr>
<td>Volkswagen Finance</td>
<td>59,20932894</td>
<td>592,093</td>
</tr>
<tr>
<td>Totale</td>
<td>372,9781192</td>
<td>3,729,781</td>
</tr>
</tbody>
</table>

That mean that starting from an initial capital of 500 millions euro, equally divided into 5 corporate bonds, we have to spend 3,729,780 euro to carry out an hedging strategy against the default of the 5 different reference entities.
Comparing the two strategies: FTD

Applying an hedging strategy based on a first-to-default option we can see different prices that depend on the correlation level between the names included in the basket:

<table>
<thead>
<tr>
<th>Correlation</th>
<th>Price in bps.</th>
<th>Price in euro (1,000)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>888,5789389</td>
<td>8885,789</td>
</tr>
<tr>
<td>0.1</td>
<td>768,3775278</td>
<td>7683,775</td>
</tr>
<tr>
<td>0.2</td>
<td>665,9541116</td>
<td>6659,541</td>
</tr>
<tr>
<td>0.3</td>
<td>577,6528318</td>
<td>5776,528</td>
</tr>
<tr>
<td>0.4</td>
<td>500,7517336</td>
<td>5007,517</td>
</tr>
<tr>
<td>0.5</td>
<td>433,1836945</td>
<td>4331,837</td>
</tr>
<tr>
<td>0.6</td>
<td>373,3513632</td>
<td>3733,514</td>
</tr>
<tr>
<td>0.7</td>
<td>320,0012821</td>
<td>3200,013</td>
</tr>
<tr>
<td>0.8</td>
<td>272,1363056</td>
<td>2721,363</td>
</tr>
<tr>
<td>0.9</td>
<td>228,9534193</td>
<td>2289,534</td>
</tr>
<tr>
<td>0.99</td>
<td>193,5490871</td>
<td>1935,491</td>
</tr>
<tr>
<td>1</td>
<td>189,7987829</td>
<td>1897,988</td>
</tr>
</tbody>
</table>
Conclusions

- The model reacts correctly to the variation of the correlation value (negative correlation spread tends to high level)

- However, it requires some arbitrary assumption:
  - Identical credit quality for the lenders;
  - Only one systematic factor.

**OBJECTIVE of my research**

- Improve the pricing model using:
  - Stochastic individual hazard rates;
  - Correlating survival times using Copula functions.